Note on C. S. Peirce's Paper on "A Quincuncial Projection of the Sphere."

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In the second volume of this journal* Professor C. S. Peirce called attention to a very elegant representation of the sphere on the plane by means of the function $\operatorname{cn}\left(z, \varkappa = \frac{1}{\sqrt{2}}\right)$. Let θ, l, p be the longitude, latitude, and north polar distance resp. of a point P on the sphere. If $\zeta = \xi + i\eta$ be the stereographic projection of this point on the equatorial plane ("ζ-plane"), we have

$$\zeta = \rho e^{\iota\theta} = \tan\frac{p}{2}\,.\,e^{i\theta}.$$

Let now $\zeta = \operatorname{cn}\left(z, \frac{1}{\sqrt{2}}\right)$; the ζ-plane and thus the sphere itself is conformally represented on the "z-plane." Being given ζ, it is not difficult to find formulæ for determining the coordinates of z and thus follow the movements of P in the z-plane.

The formulæ given by Prof. Peirce for this purpose are

$$x_\kappa = \tfrac{1}{2} F(\phi), \tag{1}$$

where x_κ is *one* of the coordinates of $z = x + iy$,

$$\cos^2\phi = \frac{\sqrt{1 - \cos^2 l \cos^2\theta} - \sin l}{1 + \sqrt{1 - \cos^2 l \cos^2\theta}}, \tag{2}$$

and as usual

$$F(\phi) = \int_0^\phi \frac{d\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}}.$$

There seems to be an error in this determination, however, as may be seen in

*C. S. Peirce, "A Quincuncial Projection of the Sphere." *American Journal of Mathematics*, Vol. II (1879), p. 394.

taking special values of l and θ. For example, for $l=\theta=0$ we have $\zeta=1$, whence $z\equiv 0\ (\text{mod. } 4K,\ 2K(1+\iota))$, so that

$$x\equiv y\equiv 0. \tag{3}$$

The formulæ (1), (2), however, require that

$$x_\kappa\equiv\frac{K}{2}\qquad(\text{mod. } K),$$

which contradicts (3).

Expressions for x, y may be determined as follows:* Let $u=x+iy$, $v=x-\iota y$; then $u+v=2x$, $u-v=2iy$ and

$$\text{cn}\ 2x=\frac{\text{cn}\ u\ \text{cn}\ v-\text{sn}\ u\ \text{sn}\ v\ \text{dn}\ u\ \text{dn}\ v}{1-\frac{1}{2}\text{sn}^2 u\ \text{sn}^2 v}.$$

Now $\text{cn}\ u=\rho e^{\iota\theta}$, whence $\text{cn}\ v=\rho e^{-\iota\theta}$. Similarly let

$$\text{sn}\ u=\rho_1 e^{\iota\theta_1}\ ,\quad \text{dn}\ u=\rho_2 e^{\iota\theta_2},$$

so that

$$\text{sn}\ v=\rho_1 e^{-\iota\theta_1},\quad \text{dn}\ v=\rho_2 e^{-\iota\theta_2},$$

whence

$$\text{cn}\ 2x=\frac{\rho^2-\rho_1^2\rho_2^2}{1-\frac{1}{2}\rho_1^4} \tag{4}$$

and

$$\text{cn}\ 2y=\frac{1-\frac{1}{2}\rho_1^4}{\rho^2+\rho_1^2\rho_2^2}. \tag{5}$$

But

$$\rho_1^4=\frac{4(1-\cos^2\theta\cos^2 l)}{(1+\sin l)^2},\quad \rho_2^4=\frac{1-\sin^2\theta\cos^2 l}{(1+\sin l)^2}.$$

Thus introducing the angles α, β, the equations (4), (5) give

$$\left.\begin{aligned}\text{cn}\ 2x&=\frac{\cos^2 l-2\sqrt{\sin^2 l+\frac{1}{4}\cos^4 l\sin^2 2\theta}}{2\sin l+\cos^2 l\cos 2\theta}=\cos\alpha,\\ \text{cn}\ 2y&=\frac{2\sin l+\cos^2 l\cos 2\theta}{\cos^2 l+2\sqrt{\sin^2 l+\frac{1}{4}\cos^4 l\sin^2 2\theta}}=\cos\beta,\end{aligned}\right\} \tag{6}$$

whence

$$x=\tfrac{1}{2}F(\alpha),\quad y=\tfrac{1}{2}F(\beta). \tag{7}$$

The corresponding formulæ expressing ξ, η in terms of x, y are

$$\xi=\frac{\text{cn}\ x\ \text{cn}\ y}{1-\text{sn}^2 y\ \text{dn}^2 x},\quad \eta=-\frac{\text{sn}\ x\ \text{sn}\ y\ \text{dn}\ x\ \text{dn}\ y}{1-\text{sn}^2 y\ \text{dn}^2 x}. \tag{8}$$

Before computing the coordinates of z for a point P on the sphere, it is well to see what general correspondence exists between the z-plane and the sphere.

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*Cf. Richelot, "Darstellung einer beliebigen Grösse durch sinam $(u+w,\ k)$." Crelle, Vol. 45. Durège, "Theorie der elliptischen Functionen." 4th ed. Leipzig, 1887, p. 289.

Figure 1 represents the stereographic projection of the sphere on the ζ-plane. The inner circle, α, has a radius $=1$; the outer circle, β, has an infinite radius; to them correspond on the sphere resp. the equator and an infinitely small circle about the south pole. Points on the northern hemisphere are projected within α, points on the southern hemisphere without α.

The point α_0 represents the N-pole. The eight lines passing through α_0 and $\beta_1, \beta_2 \ldots$ represent the eight meridian circles whose longitudes are resp. $\theta = 0°, 45°, 90° \ldots$ For shortness I shall designate any line by its terminal points; thus $(\alpha_2 \alpha_4)$ denotes for example the arc of α terminated by α_2, α_4. Similarly $\{\ldots\}$ shall represent a surface bounded by lines of the figure passing through the points within $\{\ \}$.

Let us now turn to the correspondence between the ζ- and z-plane. The parallelogram $\Pi = \{ABCD\}$ in Fig. 2 being an elementary parallelogram of periods of the function $\operatorname{cn}\left(z, \frac{1}{\sqrt{2}}\right)$, a point of the ζ-plane is represented by two points z_1, z_2 in Π. Since $z_1 + z_2 \equiv 0 \ (\text{mod. } 4K,\ 2K(1+i))$, the points z_1, z_2 are symmetrical with respect to a_3 as a center. This shows that $\{DBC\}$ or $\{DAB\}$ represents once and only once every point in the ζ-plane, and conversely. Instead, however, of employing Π to represent the ζ-plane, we may use the square $\Sigma = \{A'B'C'D'\}$. It consists of four lesser squares $\Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4$, the first two having a_0, i_2 resp. for centers. As the point a_3 is a center of symmetry (in the afore sense) the ζ-plane is represented once only by $\Sigma' = \Sigma_1 + \Sigma_2$. Thus corresponding to a point on the sphere there exists one point only in Σ', and conversely. Further to the N-hemisphere corresponds Σ_1 and to the S-hemisphere corresponds Σ_2; the equator is thus represented by the perimeters a, a' of Σ_1, Σ_2.

The point a_0 represents the N-pole, i_2 the S-pole. In general, corresponding points in the two planes are marked by the same letters and suffixes; points in the ζ-plane bearing Greek letters, and those of the z-plane, Latin. Finally, when ζ moves on any line marked in Fig. 1, z moves in Fig. 2 on corresponding lines. Between the points of a surface σ enclosed by such a path of ζ and the points of the corresponding surface s in the z-plane, there exists $1-1$ correspondence. In particular, the squares Σ_1, Σ_2 are divided into 16 triangles t, as e. g. $\{a_0\, a_1\, a_8\}$, to each of which corresponds an $\frac{1}{8}$ of a hemisphere. The proof of these statements I will illustrate for one or two cases.

For example, to see that when z describes $(a_0\, a_7)$, ζ describes continuously

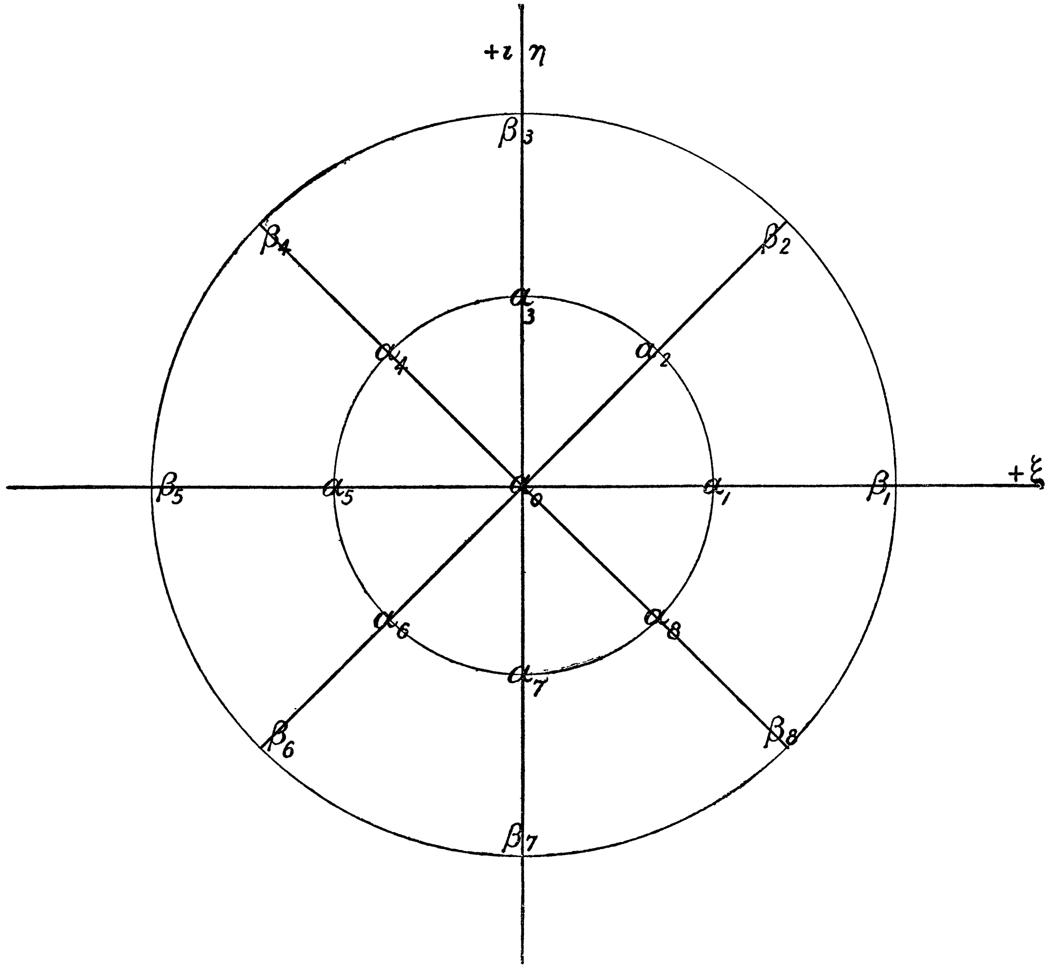

FIG. 1.—ζ-plane.

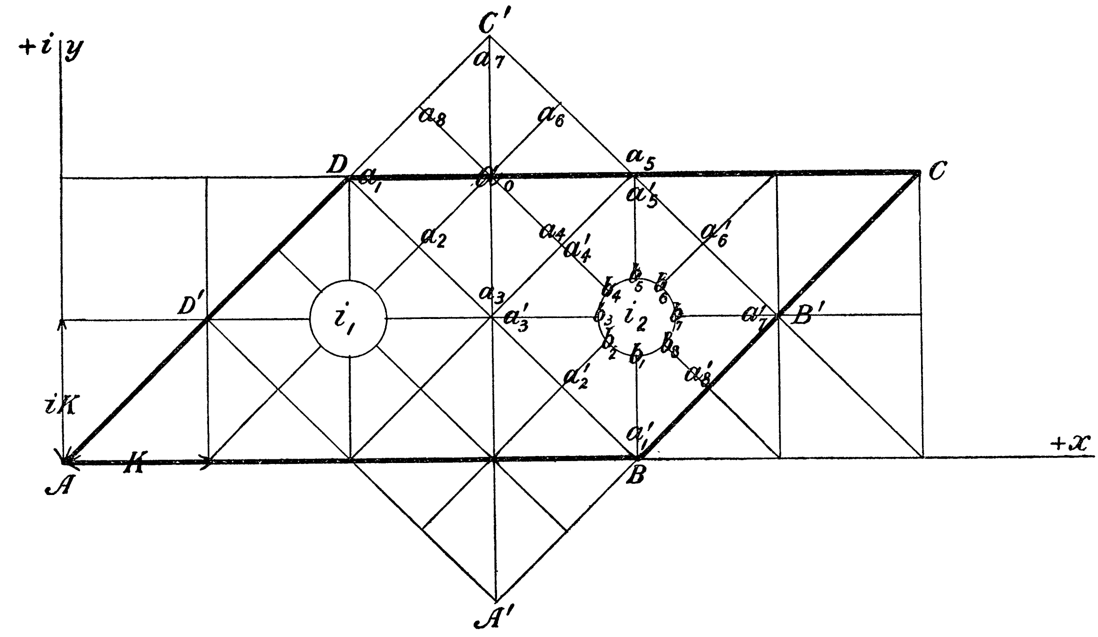

FIG. 2.—z-plane.

and without returning on itself $(\alpha_0\, \alpha_7)$, we set $z = K + iy$, $0 \gtrless y \gtrless K$. Then

$$\zeta = \operatorname{cn}(K + iy) = -\varkappa \frac{\operatorname{sn} iy}{\operatorname{dn} iy} = -\iota\varkappa \frac{\operatorname{sn} y}{\operatorname{dn} y}, \quad \varkappa = \frac{1}{\sqrt{2}}.$$

For $y = 0$, $\zeta = 0$; for $y = K$, $\zeta = -i$. As ζ is a continuous function of y, ζ moves continuously along $(\alpha_0\, \alpha_7)$; further, it cannot return on itself, for then a point ζ on $(\alpha_0\, \alpha_7)$ would be twice represented on $(a_0\, a_7)$.

To establish the correspondence between α and a, a', let us show for example that $(a_1\, a_7)$ corresponds to $(\alpha_1\, \alpha_7)$. Here $z = x(1 + i)$ and thus

$$\zeta = \operatorname{cn}(x + ix) = \frac{\operatorname{cn}^2 x - i \operatorname{sn}^2 x \operatorname{dn}^2 x}{1 - \operatorname{sn}^2 x \operatorname{dn}^2 x}, \;\therefore\; |\zeta| = 1,$$

so that ζ describes continuously and without turning the quadrant $(\alpha_1\, \alpha_7)$ while z moves on $(a_1\, a_7)$.

In a similar manner we can establish the correspondence between the medial lines $(a_2\, a_6)$, $(a_4\, a_8)$ of the square Σ_1 and the diameters $(\alpha_2\, \alpha_6)$, $(\alpha_4\, \alpha_8)$ of the circle α. This can also be readily shown by employing (8). Their quotient, namely, gives

$$\tan\theta = -\frac{\operatorname{sn} x \operatorname{sn} y \operatorname{dn} x \operatorname{dn} y}{\operatorname{cn} x \operatorname{cn} y}.$$

As here $x = y + K$, $\tan\theta = 1$, so that $(a_2\, a_6)$ corresponds to $(\alpha_2\, \alpha_6)$. The correspondence of the circles b, β is thus shown. For points in the vicinity of i_2, $z = \iota K + z'$ and

$$\begin{aligned}\zeta = \operatorname{cn}(\iota K + z') &= -\frac{\iota \operatorname{dn} z'}{\varkappa \operatorname{sn} z'} \\ &= \frac{1}{z'}(a + bz'^2 + \ldots.),\end{aligned}$$

which shows that while z describes a small circle in positive sense about i_2, ζ describes an infinitely large circle in negative sense. An inspection of the relative position of the triangles t, τ leads one to suspect that the diagonals and medial lines of Σ_1, Σ_2 are lines of symmetry; that is, if z_1, z_2 be two points in z-plane situated symmetrically in respect to such a line, then ζ_1, ζ_2 are symmetrical with respect to the corresponding line in the ζ-plane. That this is so can be illustrated on the medial $(a_2\, a_6)$. For if in (8) we replace x by $K + y$ and y by $x - K$, the expressions for ξ, η interchange. For example, the expression

for ξ becomes

$$\frac{\operatorname{cn}(K+y)\operatorname{cn}(x-K)}{1-\operatorname{dn}^2(K+y)\operatorname{sn}^2(x-K)}=\varkappa^2\frac{\operatorname{sn}x\operatorname{sn}y\operatorname{dn}x\operatorname{dn}y}{\operatorname{dn}^2x\operatorname{dn}^2y-\varkappa^2\operatorname{cn}x}=\eta.$$

In the same way we show that the diagonal $(a_3 a_7)$ is an axis of symmetry. For if $z_1=K+a+ib$ and $z_2=K-a+ib$ be two such points, we have

$$\zeta_1=\operatorname{cn}(K+a+ib)=-\varkappa\frac{\operatorname{sn}(a+ib)}{\operatorname{dn}(a+ib)}$$

$$=-\varkappa\frac{\operatorname{sn}a\operatorname{dn}b+i\operatorname{sn}b\operatorname{cn}a\operatorname{cn}b\operatorname{dn}a}{\operatorname{cn}b\operatorname{dn}a\operatorname{dn}b-\iota\varkappa^2\operatorname{sn}a\operatorname{sn}b\operatorname{cn}a}=\frac{A+iB}{C-\iota D}$$

$$=\frac{AC-BD+i(BC+AD)}{C^2+D^2}=\xi+i\eta.$$

Replacing a by $-a$, A and D become $-A$, $-D$ and

$$\zeta_2=-\xi+i\eta.$$

We may form a good idea of the representation of the parallels and meridians in the z-plane by considering the expression for the magnification

$$m=\left|\frac{dz}{dp}\right|=\left|\frac{dz}{d\zeta}\cdot\frac{d\zeta}{dp}\right|.$$

As $\dfrac{d\zeta}{dp}=\dfrac{1}{1+\sin l}$ and $\dfrac{dz}{d\zeta}=\dfrac{1}{\rho_1\rho_2}$,

$$m=\frac{1}{\sqrt{2}}\,\frac{1}{\sqrt[4]{\sin^2 l+\frac{1}{4}\cos^2 l\sin^2 2\theta}}.$$

This shows that the magnification m is greatest at the equator and least at the poles; also that along a parallel m has a minimum for $\theta=45°$, $135°$ .... and a maximum for $\theta=0°$, $90°$ .... We have already seen that the representation is conform in the vicinity of the S-pole. The same being true for the N-pole, the parallels are approximately represented in the z-plane for some distance from the poles by circles. As however they approach the equator, the above considerations show that they take on square-like forms with rounded corners. As the representation is in general conform, the z-plane meridians everywhere cut the just described parallels at right angles, so that as they depart from the lines $(a_2 a_6)$, etc., they bend inward toward the same.

The correspondence of the planes ζ and z being now pretty accurately established, we may employ the formulæ (6), (7) with advantage. The foregoing considerations show that we need to compute α, β only for values of θ lying

between 0° and 45°. We may, if we like, use only one of the angles α, β, in which case we must take θ between 0° and 90°. As an illustration of their use I append the following table for $l = 5°$:

θ	α	β	x	y	$\frac{x}{K}$	$\frac{y}{K}$	$1-\frac{x}{K}$
0	45°29	0	.4181	0	.2255	0	.7745
5	49 32	21°28	.4593	.1895	.2477	.1022	.7523
15	63 10	47 5	.6065	.4341	.3271	.2124	.6729
45	85 0	85 0	.9887	.8654	.5333	.4668	.4667

It will be noticed that although the formula given by Prof. Peirce for computing the coordinates of z is incorrect, the last two columns of the above table agree with the results given by him.

The representation afforded by $\zeta = \operatorname{cn}\left(z, \frac{1}{\sqrt{2}}\right)$ is everywhere conform except for certain points for which $\frac{d\zeta}{dz}$ becomes 0 or ∞, that for the corners of Σ_1, Σ_2 and i_2. We have already seen that the representation is conform at i_2, a fact illustrated by the diagonals and medial lines of Σ_2.

For the other points, however, the representation is not conform. To take an example a_1. For its vicinity, $\zeta = \operatorname{cn} z = (1 + az^2 + bz^4 + \ldots.)$, and thus

$$\zeta - 1 = z^2 (a + bz^2 + \ldots.),$$

which shows that two lines in the z-plane meeting under the angle ϕ meet under the angle 2ϕ in the ζ-plane. Thus $(a_1 a_2)$, $(a_1 a_8)$ make an angle of 90° in the z-plane, while in the ζ-plane they meet under an angle of 180°. Similarly $(a_1 a_0)$, $(a_1 a_2)$ meet under the angle 45° in z-plane, and under the angle 90° in the ζ-plane. At all the corners of Σ_1, Σ_2 the distortion of angles is double.

The function $\zeta = \operatorname{cn}\left(z, \frac{1}{\sqrt{2}}\right)$ possesses then the very remarkable property

of representing in 1—1 correspondence the interior of the square Σ_1 by the interior of a circle of unit radius about the origin of the ζ-plane. Only at the corners does this representation cease to be conformal.

That such a function existed was discovered by Schwarz* while searching to determine a function, under certain simple conditions, to illustrate Riemann's theorem† that it is possible in one way only to represent conformally a simply connected surface T on a circle so that to the center corresponds any point in the interior, and to a point on the circumference any point on the edge of T. For the case of a square whose corners were $\pm K$, $\pm \iota K$, Schwarz arrived at the function $\zeta = \text{sn}(u, i)$, which may also be written $\zeta = \text{cn}\left(z, \frac{1}{\sqrt{2}}\right)$, where $z = K - \sqrt{2}\,u$. This relation enables us to deduce all properties of $\text{cn}\left(z, \frac{1}{\sqrt{2}}\right)$ immediately from those of $\text{sn}(u, i)$, or conversely.

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* Schwarz, Crelle, vol. 70 (1869), p. 105–120; also Gesam. Math. Abh., vol. II, p. 65: Ueber einige Abbildungsaufgaben.

† Riemann, Gesam. Werke, p. 40. Grundlagen für eine allgemeine Theorie der Functionen einer complexen Verändlichen.